

L29 8.3 Powers and products of Trigonometric functions (三角函數的次數和乘積)

Part I: For $\sin x$ and $\cos x$

Part II: Other trigonometric functions

$$\int e^{ax} \cos(bx) dx = ? \quad \text{類似} \int e^x \cos(x) dx = ?$$

By the way~定義數學的約定，定義是數學的起家，規定、準則。

§ 8.3 Powers and products of Trigonometric functions

Part I : For $\sin x$ and $\cos x$

$$\text{Goal: } \int \sin x^m \cos^n x dx = ?$$

case I : m or n is odd, say $m=2k+1$.

Q: m 跟 n 什麼樣的狀況 substitution 用的上？

A: m 跟 n 有一個是單數。

$$\int \sin x^{2k+1} \cos^n x dx = \int \sin x (1 - \cos^2 x)^k \cos^n x dx \quad \cos x \text{ 的多項式}$$

Let $u = \cos x$, then $du = -\sin x dx$

- $\int (u \text{ 的多項式}) du = \text{積得出來}$

case II : m and n are even.

Using the following identities to reduce the power

$$\sin x \cos x = \frac{1}{2} \sin 2x, \quad \sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\text{case III : } \int \sin(mx) \cos(nx) dx, \int \sin(mx) \sin(nx) dx, \int \cos(mx) \cos(nx) dx. (m \neq n)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

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eg.

$$\textcircled{1} \int \cos^2 dx =$$

$$= \frac{1 + \cos(2x)}{2} = \frac{x}{2} + \frac{1}{4} \sin(2x) + C$$

$$\textcircled{2} \int \sin^2 x \cos^5 dx =$$

$$= \int \sin^2 x (1 - \sin^2)^2 \cos x dx = \int (\sin^2 x - 2\sin^4 x + \sin^6 x) \cos x dx = \frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C$$

$$\textcircled{3} \int \sin^2 x \cos^2 dx =$$

$$= \int \left[\frac{1}{2} \sin(2x) \right]^2 dx = \frac{1}{4} \int \frac{1 - \cos(4x)}{2} dx = \frac{1}{8} x - \frac{1}{32} \sin(4x) + C$$

$$\textcircled{4} \int \sin^5 x dx =$$

$$= \sin x (1 - \cos^2)^2 dx$$

$$\textcircled{5} \int \sin^4 x \cos^2 x dx =$$

$$= \int \frac{1 - \cos(2x)}{2} \cdot \left[\frac{1}{2} \sin(2x) \right]^2 dx = \frac{1}{8} \int \sin^2(2x) - \cos(2x) \sin^2(2x) dx$$

$$= \frac{1}{8} \int \frac{1 - \cos(4x)}{2} dx - \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{3} \sin^3(2x) = \frac{1}{16} x - \frac{1}{64} \sin(4x) - \frac{1}{48} \sin^3(2x) + C$$

$$\textcircled{6} \int \sin(5x) \cos(4x) dx =$$

$$\frac{1}{2} \int \sin(9x) + \sin(x) dx = -\frac{\cos(9x)}{18} - \frac{\cos x}{2} + C$$

Remark: $\int \sin^n x dx$, $\int \cos^n x dx$

$$\int \sin^n x dx = \int \sin^{n-1} \cdot \sin x dx = -\sin^{n-1} \cos x + (n-1) \int \cos^2 x \cdot \sin^{n-2} dx$$

$$= -\sin^{n-1} \cos x + (n-1) \int \sin^{n-2} dx - (n-1) \int \sin^n dx$$

$$\Rightarrow \frac{-1}{n} \sin^{n-1} \cos x + \frac{(n-1)}{n} \int \sin^{n-2} dx \quad \text{apply this equation for } n-2.$$

Ex:P415(2.8.16.27.29.33.34)

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Part: Other trigonometric functions

$$(I) \int \tan^n x dx, (\int \cot^n x dx)$$

Q: 能不能製造出 $\sec^2 x$? A: 抽出 $\tan^2 x$ 轉換($\sec^2 x - 1$)

$$\int \tan^n x dx = \int \tan^{n-2} (\sec^2 x - 1) dx = \int \tan^{n-2} \sec^2 x dx - \int \tan^{n-2} x dx$$

重複一直作到次數為 1 或 0

eg. $\int \tan^6 x dx$

$$= \int \tan^4 x (\sec^2 x - 1) dx = \frac{\tan^5 x}{5} - \int \sec^2 x (\sec^2 x - 1) x dx$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \int (\sec^2 x - 1) x = \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x + C$$

$$(II) \int \sec^n x dx, (\int \csc^n x dx)$$

Q: 能不能製造出 $\sec x \tan x$? A: 不能, 只能製造出 $\sec^2 x \tan^2 x$ (substitution 不能用)

If n is even, say $n=2k$

$$\int \sec^{2k} x dx = \int \sec^2 x (\tan^2 x + 1)^{k-1} dx = \int (u^2 + 1)^{k-1} du$$

If n is odd, say $n=2k+1$

Q: 要拆成怎樣? A: $\sec^{2k-1} x$ 和 $\sec^2 x$

$$\int \sec^{2k+1} x dx = \int \sec^{2k-1} x \sec^2 x dx = \sec^{2k-1} x \tan x - \int (2k-1) \sec^{2k-2} x \sec x \tan x \cdot \tan x dx$$

$$= \sec^{2k-1} x \tan x - (2k-1) \int \sec^{2k+1} x dx + (2k-1) \int \sec^{2k-1} x dx$$

$$\Rightarrow \int \sec^{2k+1} x dx = \frac{1}{2k} \tan x \sec^{2k-1} x + \frac{2k-1}{2k} \int \sec^{2k-1} x dx$$